Biased estimates in the trajectory tracking problem: the determination of the lower bound of their accuracy

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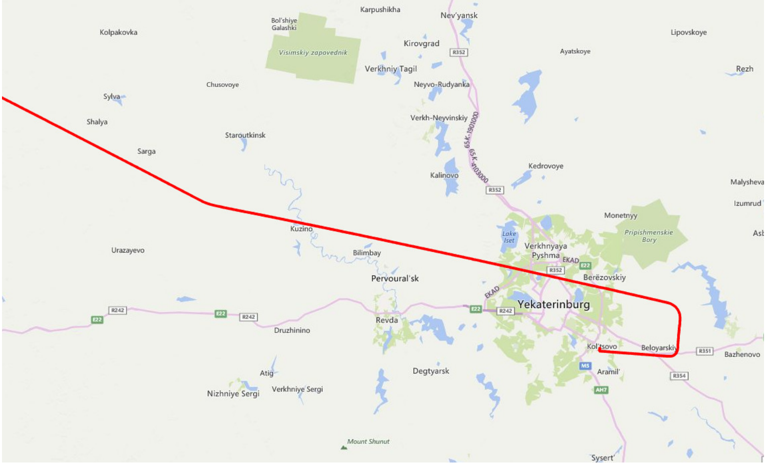
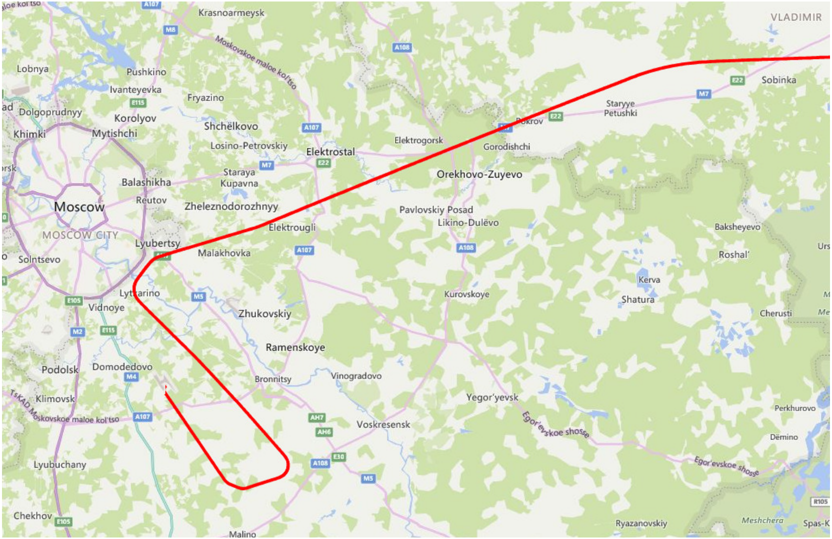
Planar Trajectories of Aircrafts

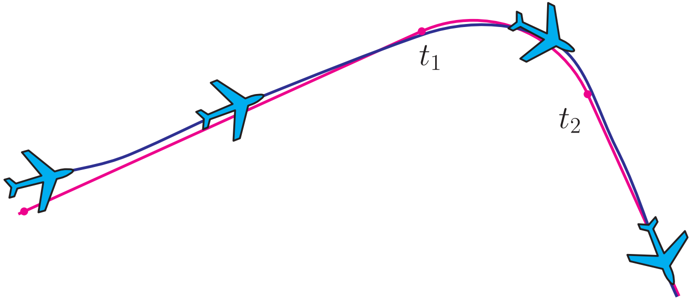


Planar Trajectories of Aircrafts



Planar Trajectories of Aircrafts

Trajectories of aircrafts are close to straight lines and circles.



Planar Trajectories of Aircrafts

Approximate planar dynamics:

$$\begin{cases} \dot{x}_N = v \cos \varphi \,, \\ \dot{x}_E = v \sin \varphi \,, \\ \dot{\varphi} = a^{ort} / v \,, \\ \dot{v} = a^{tng} \,. \end{cases}$$

- x_N, x_E are the Cartesian coordinates in plane;
- v, φ are the aircraft speed and path angle;
- a^{tng}, a^{ort} are the tangential and orthogonal accelerations, (unknown controls!)

Typical motion types:

- $a^{ort}(t) = 0$, $a^{tng}(t) = 0$ — constant velocity (CV);
- $a^{ort}(t) = \text{const}$, $a^{tng}(t) = 0$ — coordinated turn (CT);
- $a^{ort}(t) = 0$, $a^{tng}(t) = \text{const}$ — constant acceleration (CA).

Measurements

Measureable part of the phase vector

$$x = \begin{bmatrix} x_N \\ x_E \end{bmatrix}.$$

Measurement vector

$$z = \begin{bmatrix} z_N \\ z_E \end{bmatrix} = \begin{bmatrix} x_N \\ x_E \end{bmatrix} + \begin{bmatrix} w_N \\ w_E \end{bmatrix} = x + w\,.$$

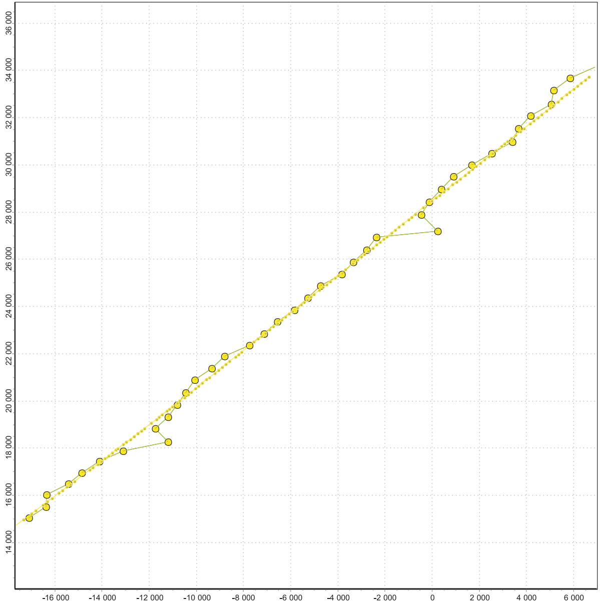
Measurements are made at discrete time instants

$$z^i = x(t^i) + w^i\,, \qquad t^i \in \{t^1, t^2, t^3, \ldots, t^n\} =: T^n\,.$$

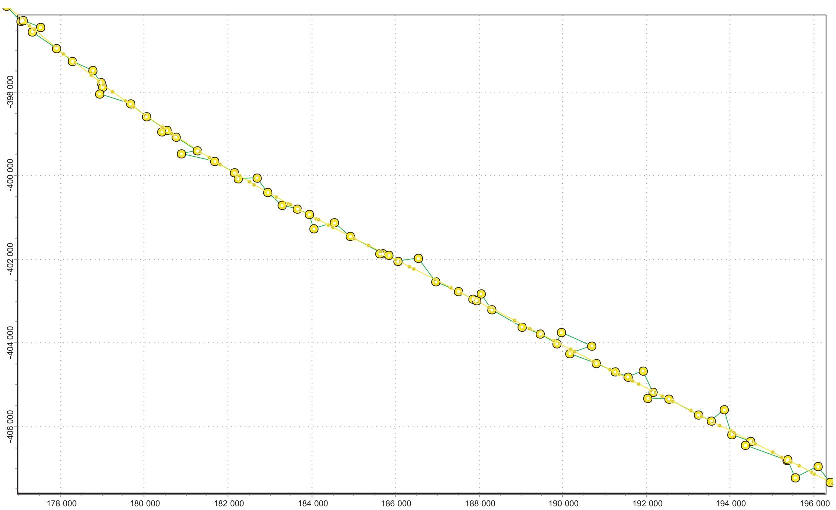
Measurement model is simple:

$$w^i \sim \mathcal{N}(0, W^i)$$

Measurements



Measurements



Trajectories

Class $\mathcal{X}^{\theta}_{switch}$: trajectories

$$x(t) = x\big(t; \theta_0, u_{[t_0,t)}\big)$$

with different initial states

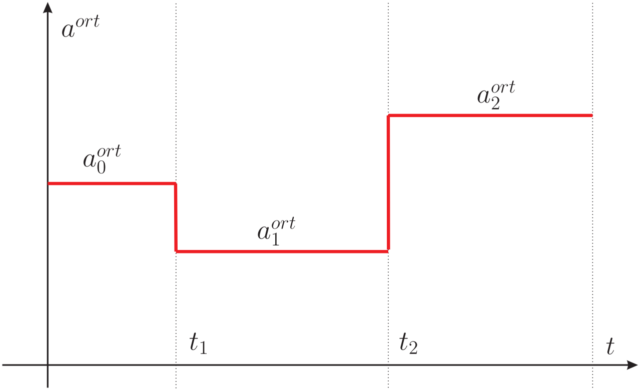
$$\theta_0 = \begin{bmatrix} x_N(t_0) \\ x_E(t_0) \\ \varphi(t_0) \\ v(t_0) \end{bmatrix}$$

and control functions

$$u_{[t_0,t)} = \left\{ \begin{bmatrix} a^{ort}(t) \\ a^{tng}(t) \end{bmatrix} \ : \quad t \in [t_0, t) \right\} .$$

Trajectories

Piecewise control functions



Vector of all parameters of the trajectory:

$$\theta = \left[\theta_0{}^\mathsf{T} \quad a_0^{ort} \quad a_0^{tng} \quad t_1 \quad a_1^{ort} \quad a_1^{tng} \quad t_2 \quad a_2^{ort} \quad a_2^{tng} \quad \ldots\right]^\mathsf{T}$$

Filtration Problem

Our aim is to make an estimate $\hat{x}^n$ close to $x(t^n;\theta)$.
We have a history of measurements up to t^n instant

$$Z^n = \begin{bmatrix} z^{1\mathsf{T}} & z^{2\mathsf{T}} & \dots & z^{n\mathsf{T}} \end{bmatrix}^{\mathsf{T}} .$$

The estimator $\hat{x}^n$ is a response to Z^n:

$$\hat{x}^n = \hat{x}^n(Z^n) .$$

Performance criterion is formulated using

$$\mathbf{E}\left\{ \left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)\left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)^{\mathsf{T}} \right\},$$

for example

$$\mathrm{tr}\left\{ \mathbf{E}\left\{ \left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)\left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)^{\mathsf{T}} \right\} \right\}$$

Filtration Problem

- *Gustafsson, F.* Adaptive filtering and change detection. Wiley: 2000.
- *Li, X. R., Jilkov, V.* A survey of maneuvering target tracking. Part IV: Decision-based methods // Proc. SPIE. 2002. Vol. 4728, pp. 511–534.
- *Li, X. R., Jilkov, V.* A survey of maneuvering target tracking. Part V: Multiple-model methods // IEEE Trans. on AES. 2005. No. 4, pp. 1255–1321.
- *Bar-Shalom, Y., Li, X. R., Kirubarajan, T.* Estimation with Application to Tracking and Navigation: Theory, Algorithms, and Software. New York: Wiley, 2001.
- *Blom, H., Bar-Shalom, Y.* The interacting multiple model algorithm for systems with Markovian switching coefficients // IEEE Transactions on Automatic Control, Vol. 33, No. 8, 1988, pp. 780–783.
- *Gustafsson, F., et al.* Particle filters for positioning, navigation, and tracking // IEEE Transactions on Signal Processing, 2002. No. 50 (2), pp. 425–437.

Lower Bound of MSE

There exsists a way to assess a lower bound of «accuracy» of estimates without explicit construction of the best estimate.

Cauchy-Schwartz inequality:

$$\langle \varphi, \psi \rangle^2 \leqslant \langle \varphi, \varphi \rangle \langle \psi, \psi \rangle \iff \langle \varphi, \varphi \rangle \geqslant \frac{\langle \varphi, \psi \rangle^2}{\langle \psi, \psi \rangle} \quad \Big(\langle \psi, \psi \rangle \neq 0 \Big).$$

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Replacing $\langle\varphi,\psi\rangle$
by $\mathbf{E}\{(\hat{x}^n(Z^n) - \bar{x}^n(\theta))\ \psi\}$ where $\bar{x}^n(\theta) = \mathbf{E}\{\hat{x}^n(Z^n)\}$
we get:

$$\mathbf{E}\Big\{(\hat{x}^n(Z^n) - \bar{x}^n(\theta))^2\Big\} \geqslant \frac{\mathbf{E}\{(\hat{x}^n(Z^n) - \bar{x}^n(\theta))\,\psi\}^2}{\mathbf{E}\{\psi\,\psi\}}.$$

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In vector case:

$$\mathbf{E}\left\{(\hat{x}^n(Z^n) - \bar{x}^n(\theta))\,(\hat{x}^n(Z^n) - \bar{x}^n(\theta))^{\mathsf{T}}\right\} \succcurlyeq$$
$$\succcurlyeq \mathbf{E}\left\{(\hat{x}^n(Z^n) - \bar{x}^n(\theta))\ \psi^{\mathsf{T}}\right\}\left(\mathbf{E}\left\{\psi\,\psi^{\mathsf{T}}\right\}\right)^{-1} \times$$
$$\times\, \mathbf{E}\left\{\psi\,(\hat{x}^n(Z^n) - \bar{x}^n(\theta))^{\mathsf{T}}\right\}.$$

Lower Bound of MSE

Total MSE with the bias term has the form:

$$\mathbf{E}\left\{ \left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)\left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)^{\mathsf{T}} \right\} \succcurlyeq$$

$$\succcurlyeq \mathbf{E}\left\{ \left(\hat{x}^n(Z^n) - \bar{x}^n(\theta)\right)\, \psi^{\mathsf{T}} \right\} \left(\mathbf{E}\left\{ \psi\, \psi^{\mathsf{T}} \right\} \right)^{-1} \times$$

$$\times\, \mathbf{E}\left\{ \psi \left(\hat{x}^n(Z^n) - \bar{x}^n(\theta)\right)^{\mathsf{T}} \right\} +$$

$$+ \left(\bar{x}^n(\theta) - x(t^n;\theta)\right)\left(\bar{x}^n(\theta) - x(t^n;\theta)\right)^{\mathsf{T}}.$$

Lower Bound of MSE

In Bayesian approach, there exists a distribution on θ, and the expectation means

$$\mathbf{E}\{f\} = \mathbf{E}_{\theta,Z^n}\{f\} = \int_\theta \int_{Z^n} f\,\rho(Z^n|\theta)\,dZ^n\,\tilde{\rho}(\theta)\,d\theta,$$

while, in non-Bayesian approach, the expectation concerns only disribution of measurements

$$\mathbf{E}\{f\} = \mathbf{E}_{Z^n}\{f\} = \int_{Z^n} f\,\rho(Z^n|\theta)\,dZ^n.$$

I consider non-Bayesian approach only because

I am interested in a lower bound for each particular trajectory

Lower Bound of MSE

$$\mathbf{E}\left\{\left(\hat{x}^n(Z^n) - \bar{x}^n(\theta)\right)\left(\hat{x}^n(Z^n) - \bar{x}^n(\theta)\right)^{\mathsf{T}}\right\} \succcurlyeq$$
$$\succcurlyeq \mathbf{E}\left\{\left(\hat{x}^n(Z^n) - \bar{x}^n(\theta)\right)\,\psi^{\mathsf{T}}\right\}\left(\mathbf{E}\left\{\psi\,\psi^{\mathsf{T}}\right\}\right)^{-1} \times$$
$$\times\,\mathbf{E}\left\{\psi\left(\hat{x}^n(Z^n) - \bar{x}^n(\theta)\right)^{\mathsf{T}}\right\}.$$

Different ψ's, different lower bounds:

- $\psi = \frac{1}{\rho(Z^n|\theta)}\left(\frac{\partial\rho(Z^n|\theta)}{\partial\theta}\right)^{\mathsf{T}} = \left(\frac{\partial}{\partial\theta}\log\rho(Z^n|\theta)\right)^{\mathsf{T}}$

  for Cramér-Rao lower bound (CR);

- $\psi = \frac{\rho(Z^n|\theta') - \rho(Z^n|\theta)}{\rho(Z^n|\theta)}$ $\qquad \theta'\colon\ \exists i\ x(t^i;\theta') \neq x(t^i;\theta)$

  for Hammersley-Chapman-Robbins bound (HCR).

Cramer-Rao Lower Bound. General Case

$$\psi = \frac{1}{\rho(Z^n|\theta)} \left(\frac{\partial \rho(Z^n|\theta)}{\partial \theta} \right)^{\mathsf{T}}$$

$$\mathbf{E}\Big\{ \left(\hat{x}^n(Z^n) - x(t^n;\theta)\right) \left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)^{\mathsf{T}} \Big\} \succcurlyeq$$
$$\succcurlyeq \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta} \right) I(\theta)^{-1} \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta} \right)^{\mathsf{T}} +$$
$$+ \left(\bar{x}^n(\theta) - x(t^n;\theta)\right) \left(\bar{x}^n(\theta) - x(t^n;\theta)\right)^{\mathsf{T}},$$

where

$$\bar{x}^n(\theta) = \mathbf{E}\{\hat{x}^n(Z^n)\}, \qquad I(\theta) = \sum_{i=1}^{n} \frac{\partial x(t^i;\theta)}{\partial \theta}^{\mathsf{T}} (W^i)^{-1} \frac{\partial x(t^i;\theta)}{\partial \theta}.$$

Cramer-Rao Lower Bound. General Case

$$\mathbf{E}\left\{ (\hat{x}^n(Z^n) - x(t^n;\theta)) (\hat{x}^n(Z^n) - x(t^n;\theta))^{\mathsf{T}} \right\} \succcurlyeq$$
$$\succcurlyeq \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta} \right) I(\theta)^{-1} \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta} \right)^{\mathsf{T}} +$$
$$+ (\bar{x}^n(\theta) - x(t^n;\theta)) (\bar{x}^n(\theta) - x(t^n;\theta))^{\mathsf{T}},$$

- $\bar{x}^n(\theta)$ is vector in $\mathbb{R}^2$;
- $\frac{\partial \bar{x}^n(\theta)}{\partial \theta}$ is matrix in $\mathbb{R}^{2\times d}$.

Cramer-Rao Lower Bound. General Case

$$\min_{\hat{x}^n} \mathbf{E}\left\{\left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)(\ldots)^\mathsf{T}\right\} \succcurlyeq$$

$$\succcurlyeq \min_{\hat{x}^n} \left(\left(\frac{\partial \bar{x}^n(\theta)}{\partial\theta}\right) I(\theta)^{-1} \left(\frac{\partial \bar{x}^n(\theta)}{\partial\theta}\right)^\mathsf{T} + \left(\bar{x}^n(\theta) - x(t^n;\theta)\right)(\ldots)^\mathsf{T}\right) =$$

$$= \min_{\bar{x}^n = \mathbf{E}\{\hat{x}^n\}} \left(\left(\frac{\partial \bar{x}^n(\theta)}{\partial\theta}\right) I(\theta)^{-1}(\ldots)^\mathsf{T} + \left(\bar{x}^n(\theta) - x(t^n;\theta)\right)(\ldots)^\mathsf{T}\right) \succcurlyeq$$

$$\succcurlyeq \min_{q\in\mathbb{R}^2, Q\in\mathbb{R}^{2\times d}} \left(Q\, I(\theta)^{-1} Q^\mathsf{T} + \left(q - x(t^n;\theta)\right)\left(q - x(t^n;\theta)\right)^\mathsf{T}\right)$$

CRLB, General Case. Paradox

$$\min_{\hat{x}^n} \mathbf{E}\left\{ \left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)(\ldots)^{\mathsf{T}} \right\} \succcurlyeq$$

$$\succcurlyeq \min_{\hat{x}^n} \left(\left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right) I(\theta)^{-1} \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right)^{\mathsf{T}} + \left(\bar{x}^n(\theta) - x(t^n;\theta)\right)(\ldots)^{\mathsf{T}} \right) =$$

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$$\succcurlyeq \min_{q \in \mathbb{R}^2, Q \in \mathbb{R}^{2\times d}} \left(Q\, I(\theta)^{-1} Q^{\mathsf{T}} + \left(q - x(t^n;\theta)\right)\left(q - x(t^n;\theta)\right)^{\mathsf{T}} \right)$$

$$= 0 \;!!!$$

since we can set $Q = 0$, $q = x(t^n;\theta)$.

CRLB, General Case. Paradox

This inequality is tight because constant estimates

$$\hat{x}_c^n(Z^n) = p$$

are «legal», if we consider estimates as measureable functions of Z^n (the class of measureable estimates $\mathcal{K}_m$).

$$\min_{\hat{x}^n} \mathbf{E}\Big\{(\hat{x}^n(Z^n) - x(t^n;\theta))\,(\ldots)^\mathsf{T}\Big\} \preccurlyeq$$
$$\preccurlyeq \min_{\hat{x}_c^n} \mathbf{E}\Big\{(\hat{x}^n(Z^n) - x(t^n;\theta))\,(\ldots)^\mathsf{T}\Big\} =$$
$$= \min_{p}\,(p - x(t^n;\theta))\,(p - x(t^n;\theta))^\mathsf{T} = 0$$

$$0 \preccurlyeq \min_{\hat{x}^n} \mathbf{E}\Big\{(\hat{x}^n(Z^n) - x(t^n;\theta))\,(\ldots)^\mathsf{T}\Big\} \preccurlyeq 0 \implies$$
$$\min_{\hat{x}^n} \mathbf{E}\Big\{(\hat{x}^n(Z^n) - x(t^n;\theta))\,(\ldots)^\mathsf{T}\Big\} = 0$$

CRLB, General Case. Paradox

This inequality is tight because constant estimates

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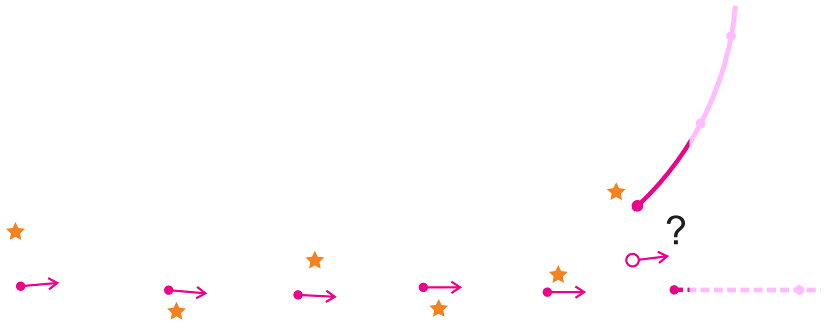
$$\min_{\hat{x}^n} \mathbf{E}\Big\{ (\hat{x}^n(Z^n) - x(t^n;\theta))\,(\dots)^{\mathsf{T}} \Big\} =$$

$$= \min_{\hat{x}_c^n} \mathbf{E}\Big\{ (\hat{x}^n(Z^n) - x(t^n;\theta))\,(\dots)^{\mathsf{T}} \Big\} =$$

$$= \min_{p} \,(p - x(t^n;\theta))\,(p - x(t^n;\theta))^{\mathsf{T}} = 0$$

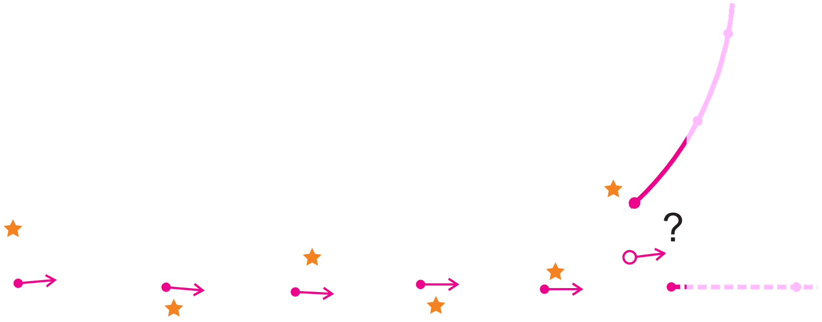
Necessarily, there is one right choice among all!

Different Trajectories



The constant estimate $\hat{x}_c^n(Z^n) = p$ is clearly not useful for trajectories where $x(t^n;\theta) \neq p$.

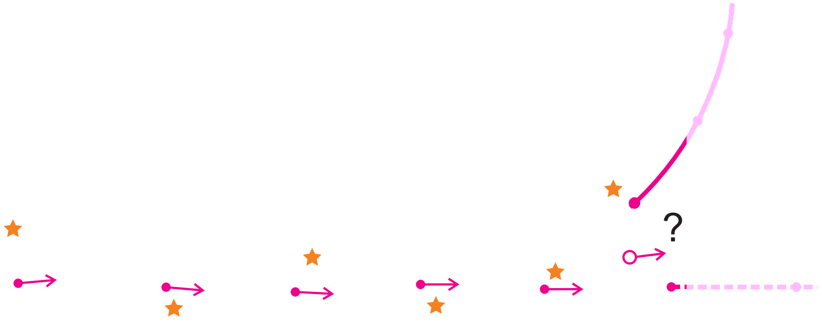
Different Trajectories



An estimate have to work for all trajectories of prescribed type.

How type of formulas can we use to set such a condition?

Unbiased Estimates



The class of unbiased estimates $\mathcal{K}_u(\mathcal{X}^{\theta}_{switch})$:

$$\mathbf{E}\{\hat{x}^n(Z^n)\} \equiv x(t^n;\theta)$$

The property of unbiasedness can depend on the trajectory class; therefore, $\mathcal{K}_u(\mathcal{X}^{\theta}_{switch})$ rather then $\mathcal{K}_u$.

CRLB, Unbiased Estimates

$$\mathbf{E}\left\{ \left(\hat{x}^n(Z^n) - x(t^n;\theta)\right) \left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)^{\mathsf{T}} \right\} \succcurlyeq$$

$$\succcurlyeq \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right) I(\theta)^{-1} \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right)^{\mathsf{T}} +$$

$$+ \left(\bar{x}^n(\theta) - x(t^n;\theta)\right) \left(\bar{x}^n(\theta) - x(t^n;\theta)\right)^{\mathsf{T}}.$$

CRLB, Unbiased Estimates

$$\mathbf{E}\left\{\left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)\left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)^{\mathsf{T}}\right\} \succcurlyeq$$
$$\succcurlyeq \left(\frac{\partial x(t^n;\theta)}{\partial\theta}\right) I(\theta)^{-1} \left(\frac{\partial x(t^n;\theta)}{\partial\theta}\right)^{\mathsf{T}}.$$

$$I(\theta) = \sum_{i=1}^{n} \frac{\partial x(t^i;\theta)}{\partial\theta}^{\mathsf{T}} (W^i)^{-1} \frac{\partial x(t^i;\theta)}{\partial\theta}.$$

CRLB, Unbiased Estimates

$$\begin{cases} \dot{x}_N = v \cos \varphi \,, \\ \dot{x}_E = v \sin \varphi \,, \\ \dot{\varphi} = a^{ort}/v \,, \\ \dot{v} = a^{tng} \,. \end{cases}$$

CRLB, Unbiased Estimates

$$\begin{cases} \dot{x}_N = v \cos \varphi \,, \\ \dot{x}_E = v \sin \varphi \,, \\ \dot{\varphi} = a^{ort} / v \,, \\ \dot{v} = a^{tng} \,. \end{cases}$$

$$\theta = \begin{bmatrix} {\theta_0}^{\mathsf{T}} & a_0^{ort} & a_0^{tng} & t_1 & a_1^{ort} & a_1^{tng} & t_2 & a_2^{ort} & a_2^{tng} & \dots \end{bmatrix}^{\mathsf{T}} .$$

CRLB, Unbiased Estimates

$$\begin{cases} \dot{x}_N = v \cos\varphi\,, \\ \dot{x}_E = v \sin\varphi\,, \\ \dot{\varphi} = a^{ort}/v\,, \\ \dot{v} = a^{tng}\,. \end{cases}$$

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$$\frac{\partial x_N(t)}{\partial a^{ort}} = \frac{\partial b}{\partial a^{ort}} \left(v(t)^2 \cos\varphi(t) - v_0^2 \cos\varphi_0 \right) + \\ + \frac{\partial c}{\partial a^{ort}} \left(v(t)^2 \sin\varphi(t) - v_0^2 \sin\varphi_0 \right) - \\ - \frac{\partial \varphi(t)}{\partial a^{ort}} v(t)^2 \left(b \sin\varphi(t) - c \cos\varphi(t) \right)$$

CRLB, Unbiased Estimates

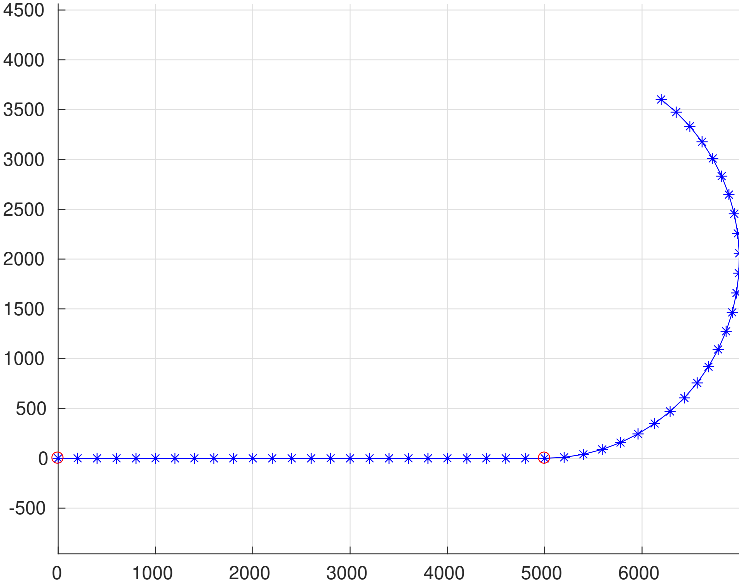
$$\theta = \begin{bmatrix} \theta_0{}^{\mathsf{T}} & a_0^{ort} & a_0^{tng} & t_1 & a_1^{ort} & a_1^{tng} & t_2 & a_2^{ort} & a_2^{tng} & \dots \end{bmatrix}^{\mathsf{T}} .$$

$$\begin{aligned} \frac{\partial x_N(t)}{\partial a^{ort}} = \frac{\partial b}{\partial a^{ort}} \left(v(t)^2 \cos\varphi(t) - v_0^2 \cos\varphi_0 \right) + \\ + \frac{\partial c}{\partial a^{ort}} \left(v(t)^2 \sin\varphi(t) - v_0^2 \sin\varphi_0 \right) - \\ - \frac{\partial \varphi(t)}{\partial a^{ort}} v(t)^2 \left(b \sin\varphi(t) - c\cos\varphi(t) \right) \end{aligned}$$

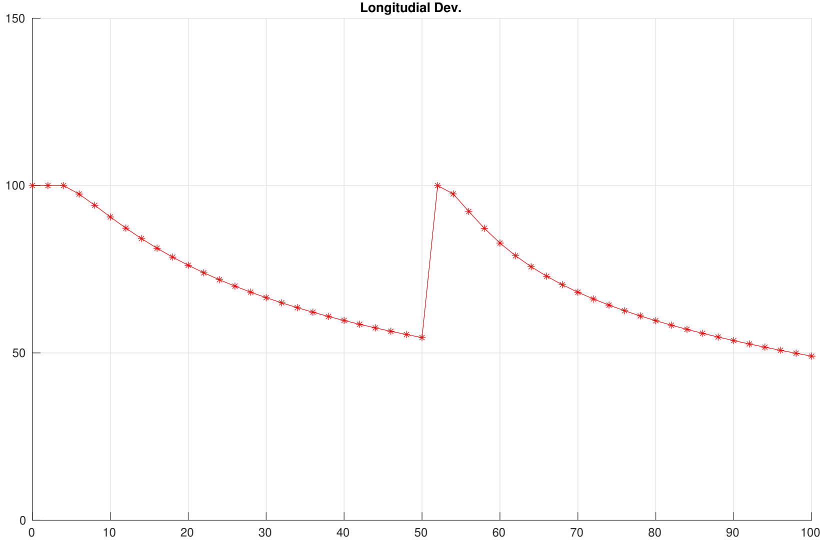
$$c = \frac{a^{ort}}{(a^{ort})^2 + (2a^{tng})^2}\,, \quad b = \frac{2a^{tng}}{(a^{ort})^2 + (2a^{tng})^2}$$

$$\frac{\partial \varphi(t)}{\partial a^{ort}} = \begin{cases} \frac{1}{a^{tng}} \ln \frac{v(t)}{v_0}, & a^{tng} \neq 0\,, \\ \frac{(t - t^0)}{v_0}, & a^{tng} = 0\,, \end{cases}$$

IMM Method vs. CRLB, Unbiased



IMM Method vs. CRLB, Unbiased



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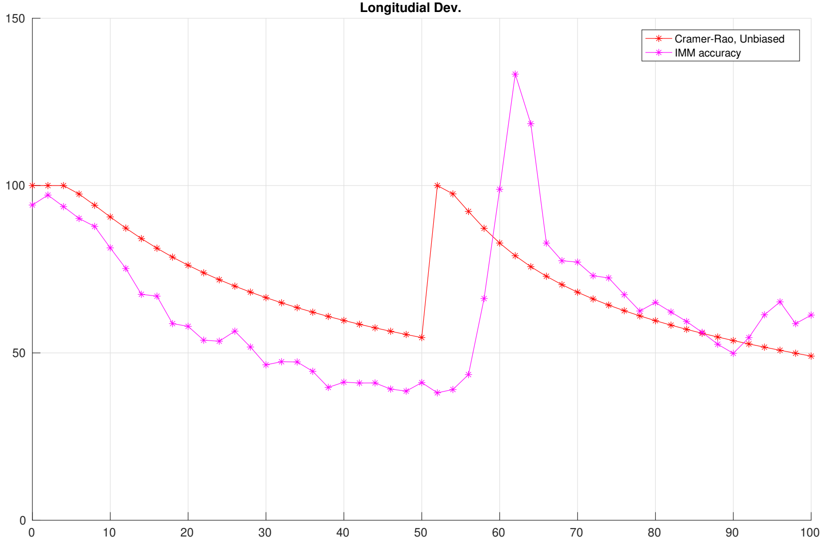

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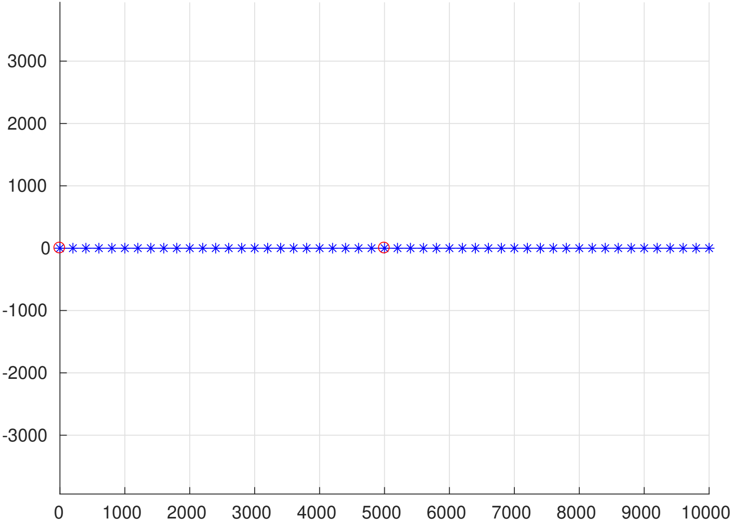
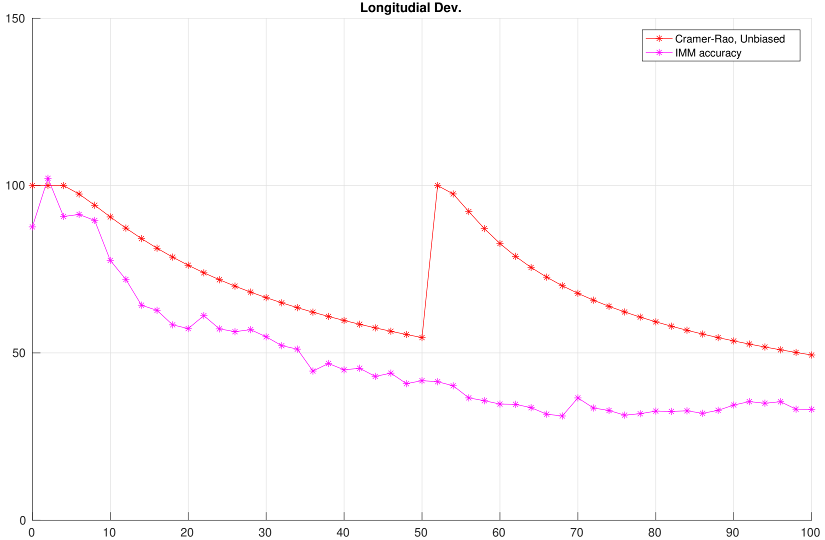
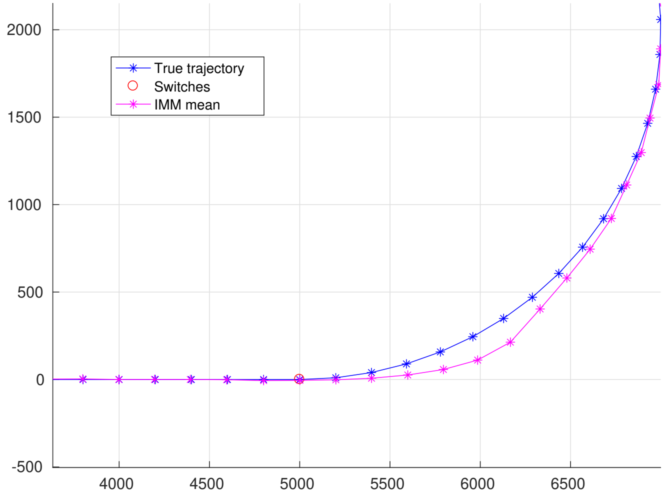


IMM Method vs. CRLB, Unbiased

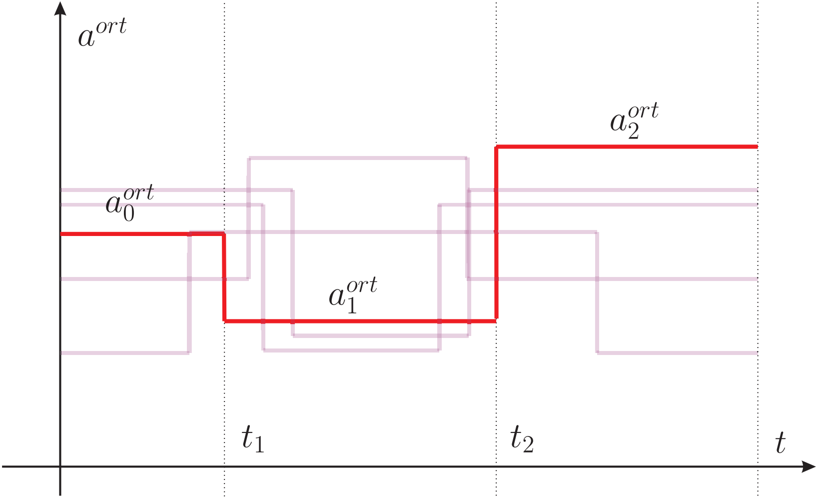


IMM Method vs. CRLB, Unbiased

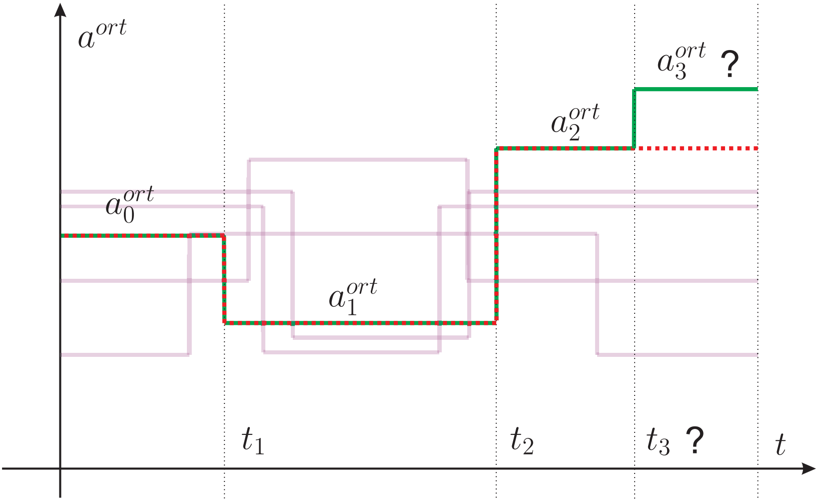
The IMM method produce biased estimates:



Trajectory variations



Trajectory variations



Hammersley-Chapman-Robbins Lower Bound

$$\mathbf{E}\Big\{\left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)\left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)^{\mathsf{T}}\Big\} \succcurlyeq$$

$$\succcurlyeq \mathbf{E}\Big\{\left(\hat{x}^n(Z^n) - \bar{x}^n(\theta)\right)\, \psi^{\mathsf{T}}\Big\}\left(\mathbf{E}\Big\{\psi\,\psi^{\mathsf{T}}\Big\}\right)^{-1} \times$$

$$\times\, \mathbf{E}\Big\{\psi\left(\hat{x}^n(Z^n) - \bar{x}^n(\theta)\right)^{\mathsf{T}}\Big\} +$$

$$+ \left(\bar{x}^n(\theta) - x(t^n;\theta)\right)\left(\bar{x}^n(\theta) - x(t^n;\theta)\right)^{\mathsf{T}}.$$

Hammersley-Chapman-Robbins Lower Bound

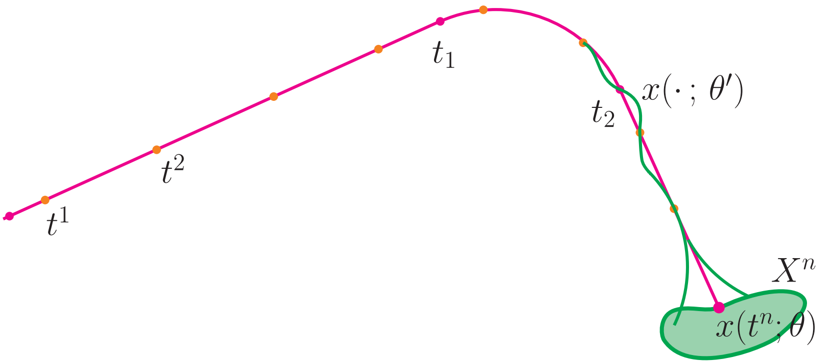
$$\forall\theta' : \ \exists i \ x(t^i;\theta') \neq x(t^i;\theta),$$

$$\mathbf{E}\left\{ \left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)(\ldots)^{\mathsf{T}} \right\} \succcurlyeq \frac{\left(x(t^n;\theta') - x(t^n;\theta)\right)(\ldots)^{\mathsf{T}}}{\mathbf{E}\left\{ \left(\frac{\rho(Z^n|\theta')}{\rho(Z^n|\theta)} - 1 \right)^2 \right\}}.$$

$$\mathbf{E}\left\{ \left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)(\ldots)^{\mathsf{T}} \right\} \succcurlyeq$$

$$\succcurlyeq \frac{\left(x(t^n;\theta') - x(t^n;\theta)\right)(\ldots)^{\mathsf{T}}}{e^{\left(\sum_{j=1}^{n} (x(t^j;\theta') - x(t^j;\theta))^{\mathsf{T}} (W^j)^{-1} (x(t^j;\theta') - x(t^j;\theta)) \right)} - 1}.$$

MSE Lower Bound for Unbiased Estimates



MSE Lower Bound for Unbiased Estimates

Statement

If

- $\hat{x}^n \in \mathcal{K}_u(\mathcal{X}^\theta_{switch})$ (i.e. $\mathbf{E}\{\hat{x}^n(Z^n)\} \equiv x(t;\theta)$),
- $\dim(X^n) = \dim z$,
- $x(t^n;\theta) \in \mathrm{int} X^n$ or
  $x(t^n;\theta) \in \partial X^n$ and ∂X^n at $x(t^n;\theta)$ is smooth

then

$$\mathbf{E}\Big\{(\hat{x}^n(Z^n) - x(t^n;\theta))\,(\ldots)^\mathsf{T}\Big\} \succcurlyeq W^n.$$

$W^n = \mathbf{Var}\{z^n\}$ is the covariation matrix of the last measurement.

Equivariant Estimates

$$\bar{x}^n(\theta) = \mathbf{E}\{\hat{x}^n(Z^n)\} \equiv x(t^n;\theta)$$

Equivariant Estimates

$$\bar{x}^n(\theta) = \mathbf{E}\{\hat{x}^n(Z^n)\} \neq x(t^n;\theta)$$

Equivariant Estimates

$$\bar{x}^n(\theta) = \mathbf{E}\{\hat{x}^n(Z^n)\} \neq x(t^n;\theta)$$

Consider affine transforms:

$$z_p^i = s_p(z^i) = B_p z^i + b_p\,, \qquad \mathcal{S}_p(Z^n) = \left\{z_p^i\ :\ i = 1, \ldots, n\right\}.$$

Equivariant Estimates

$$\bar{x}^n(\theta) = \mathbf{E}\{\hat{x}^n(Z^n)\} \neq x(t^n;\theta)$$

Consider affine transforms:

$$z_p^i = s_p(z^i) = B_p z^i + b_p\,, \qquad \mathcal{S}_p(Z^n) = \left\{z_p^i\;:\;i = 1,\ldots,n\right\}.$$

The class of equivariant estimates $\mathcal{K}_e(\mathcal{X}^\theta_{switch})$:

$$\hat{x}^n(\mathcal{S}_p(Z^n)) = s_p(\hat{x}^n(Z^n))$$

As a consequence:

Equivariant Estimates

$$\bar{x}^n(\theta) = \mathbf{E}\{\hat{x}^n(Z^n)\} \neq x(t^n;\theta)$$

Consider affine transforms:

$$z_p^i = s_p(z^i) = B_p z^i + b_p\,, \qquad \mathcal{S}_p(Z^n) = \left\{z_p^i \;:\; i = 1,\ldots,n\right\}.$$

The class of equivariant estimates $\mathcal{K}_e(\mathcal{X}_{switch}^{\theta})$:

$$\hat{x}^n(\mathcal{S}_p(Z^n)) = s_p(\hat{x}^n(Z^n))$$

As a consequence:

$$\bar{x}^n(T_p(\theta)) \equiv s_p(\bar{x}^n(\theta)),$$

where $T_p(\theta)$: $\quad \forall i \quad x(t^i; T_p(\theta)) = s_p(x(t^i;\theta))$.

Equivariant Estimates

$$\bar{x}^n(T_p(\theta)) \equiv s_p(\bar{x}^n(\theta)),$$

$$\text{where } T_p(\theta) : \quad \forall i \quad x(t^i; T_p(\theta)) = s_p(x(t^i; \theta)).$$

$$\frac{\partial}{\partial p}\,\bar{x}^n(T_p(\theta)) = \frac{\partial}{\partial p}\,s_p(\bar{x}^n(\theta)),$$

$$\Downarrow$$

Equivariant Estimates

$$\bar{x}^n(T_p(\theta)) \equiv s_p(\bar{x}^n(\theta)),$$

where $T_p(\theta)$: $\quad \forall i \quad x(t^i; T_p(\theta)) = s_p(x(t^i; \theta)).$

$$\frac{\partial}{\partial p}\,\bar{x}^n(T_p(\theta)) = \frac{\partial}{\partial p}\,s_p(\bar{x}^n(\theta)),$$

$$\Downarrow$$

$$\frac{\partial}{\partial p}\,\bar{x}^n(T_p(\theta))\Big|_{p=0} = \frac{\partial}{\partial p}\,s_p(\bar{x}^n(\theta))\Big|_{p=0},$$

Equivariant Estimates

$$\bar{x}^n(T_p(\theta)) \equiv s_p(\bar{x}^n(\theta)),$$

where $T_p(\theta)$: $\quad \forall i \quad x(t^i; T_p(\theta)) = s_p(x(t^i; \theta)).$

$$\frac{\partial}{\partial p}\, \bar{x}^n(T_p(\theta)) = \frac{\partial}{\partial p}\, s_p(\bar{x}^n(\theta)),$$

$$\Downarrow$$

$$\frac{\partial}{\partial \theta}\, \bar{x}^n(\theta)) \frac{\partial}{\partial p}\, T_p(\theta))\Big|_{p=0} = \frac{\partial}{\partial p}\, s_p(\bar{x}^n(\theta))\Big|_{p=0} =$$

$$= \frac{\partial}{\partial p}\, B_p\Big|_{p=0} \bar{x}^n(\theta) + \frac{\partial}{\partial p}\, b_p\Big|_{p=0},$$

Equivariant Estimates

$$\bar{x}^n(T_p(\theta)) \equiv s_p(\bar{x}^n(\theta)),$$

where $T_p(\theta)$: $\quad \forall i \quad x(t^i; T_p(\theta)) = s_p(x(t^i; \theta)).$

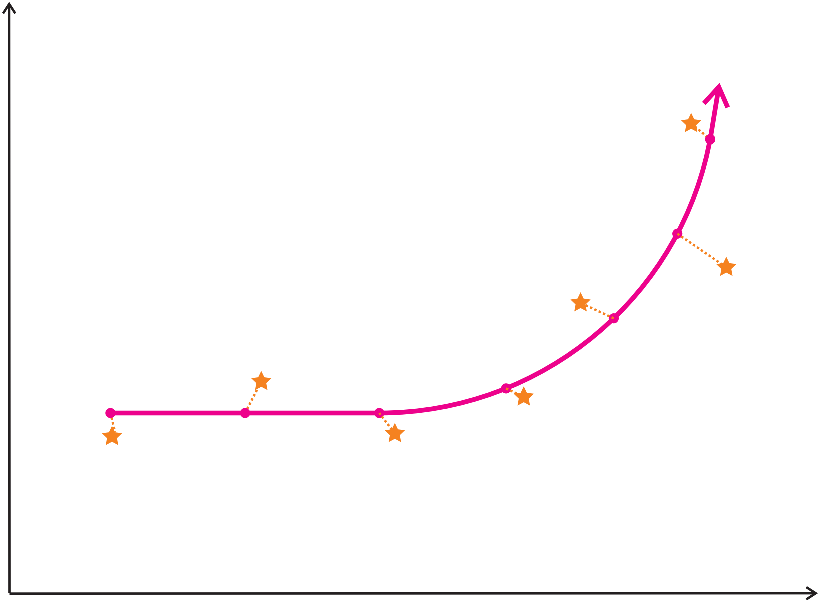
$$\frac{\partial}{\partial p}\,\bar{x}^n(T_p(\theta)) = \frac{\partial}{\partial p}\,s_p(\bar{x}^n(\theta)),$$

$$\Downarrow$$

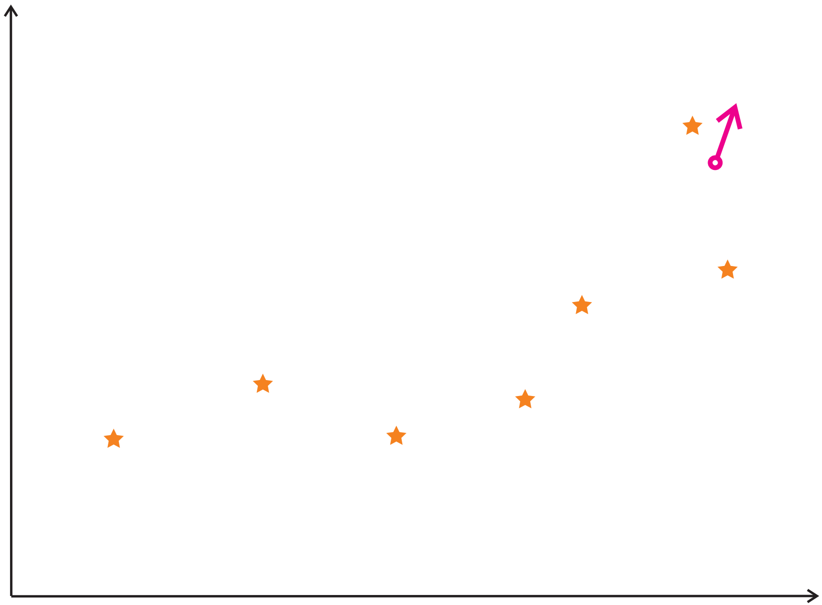
$$\frac{\partial}{\partial \theta}\,\bar{x}^n(\theta))\,\dot{T}_0(\theta) = \frac{\partial}{\partial p}\,s_p(\bar{x}^n(\theta))\Big|_{p=0} =$$

$$= \dot{B}_0\,\bar{x}^n(\theta) + \dot{b}_0,$$

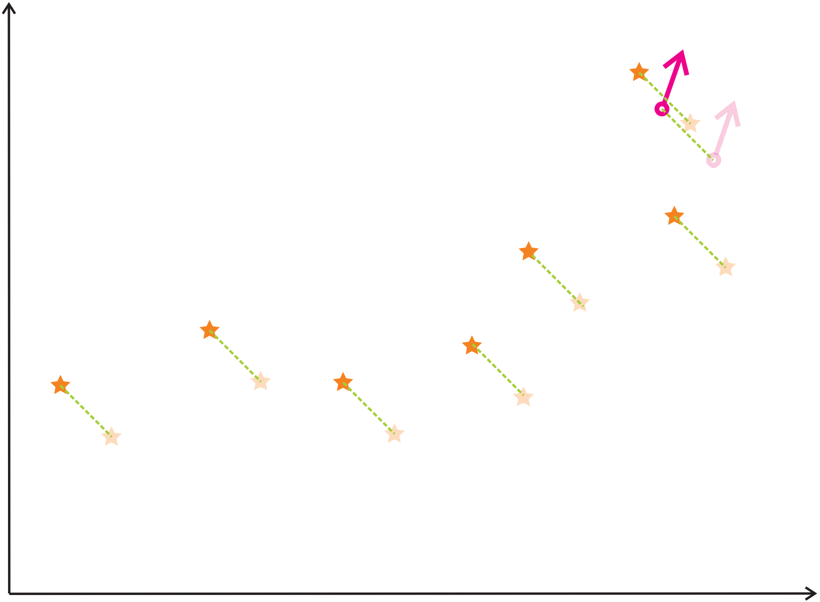
Transformations on the Plane



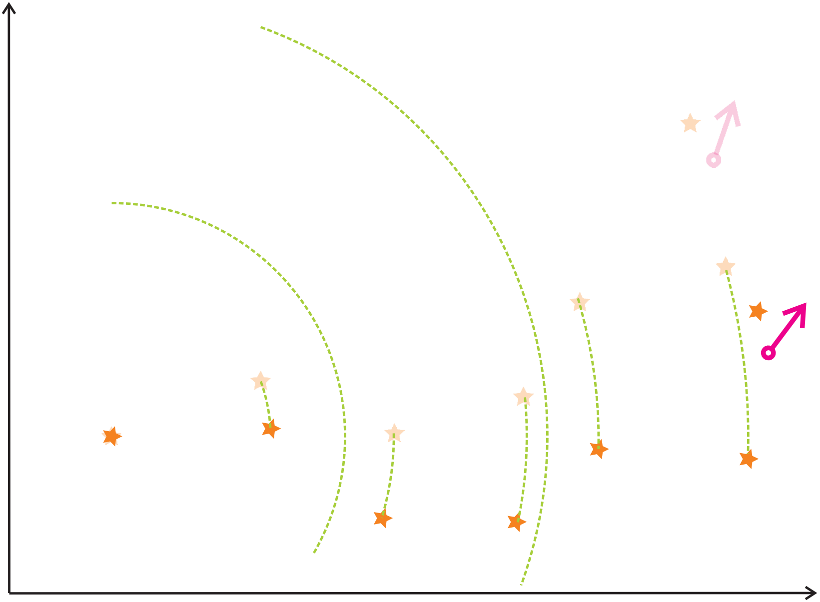
Transformations on the Plane



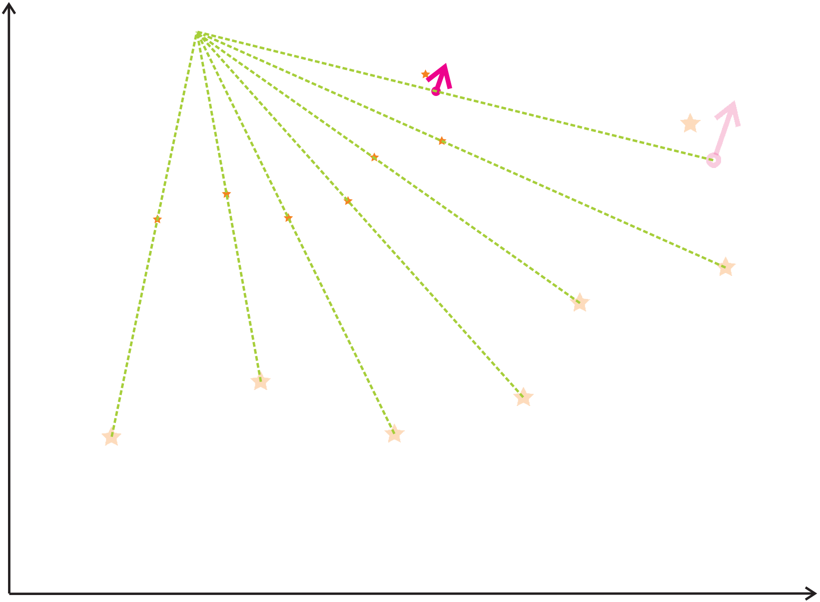
Transformations on the Plane



Transformations on the Plane



Transformations on the Plane



Equivariant Estimates. Accuracy Assessment

Cramér-Rao lower bound for general case:

$$\mathbf{E}\left\{ \left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)\left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)^{\mathsf{T}} \right\} \succcurlyeq$$
$$\succcurlyeq \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right) I(\theta)^{-1} \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right)^{\mathsf{T}} +$$
$$+ \left(\bar{x}^n(\theta) - x(t^n;\theta)\right)\left(\bar{x}^n(\theta) - x(t^n;\theta)\right)^{\mathsf{T}},$$

$$\min_{\hat{x}^n \in \mathrm{Equiv}} \mathbf{E}\left\{ \left(\hat{x}^n(Z^n) - x(t^n;\theta)\right)(\ldots)^{\mathsf{T}} \right\} \succcurlyeq$$
$$\succcurlyeq \min_{\substack{\hat{x}^n \in \mathrm{Equiv} \\ \bar{x}^n = \mathbf{E}\{\hat{x}^n\}}} \left(\left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right) I(\theta)^{-1} (\ldots)^{\mathsf{T}} + \left(\bar{x}^n(\theta) - x(t^n;\theta)\right)(\ldots)^{\mathsf{T}} \right)$$

Equivariant Estimates. Accuracy Assessment

Cramér-Rao lower bound for general case:

$$\mathbf{E}\left\{(\hat{x}^n(Z^n) - x(t^n;\theta))\,(\hat{x}^n(Z^n) - x(t^n;\theta))^{\mathsf{T}}\right\} \succcurlyeq$$

$$\succcurlyeq \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right) I(\theta)^{-1} \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right)^{\mathsf{T}} +$$

$$+\,(\bar{x}^n(\theta) - x(t^n;\theta))\,(\bar{x}^n(\theta) - x(t^n;\theta))^{\mathsf{T}},$$

$$\min_{\hat{x}^n \in \mathrm{Equiv}} \mathbf{E}\left\{(\hat{x}^n(Z^n) - x(t^n;\theta))\,(\ldots)^{\mathsf{T}}\right\} \succcurlyeq$$

$$\succcurlyeq \min_{\substack{\hat{x}^n \in \mathrm{Equiv} \\ \bar{x}^n = \mathbf{E}\{\hat{x}^n\}}} \left(\left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right) I(\theta)^{-1} (\ldots)^{\mathsf{T}} + (\bar{x}^n(\theta) - x(t^n;\theta))\,(\ldots)^{\mathsf{T}}\right) \succcurlyeq$$

$$\succcurlyeq \min_{\substack{q \in \mathbb{R}^2, Q \in \mathbb{R}^{2\times d} \\ q, Q:\ \hat{x}^n \in \mathrm{Equiv}}} \left(Q\, I(\theta)^{-1} Q^{\mathsf{T}} + (q - x(t^n;\theta))\,(q - x(t^n;\theta))^{\mathsf{T}}\right)$$

Equivariant Estimates. Accuracy Assessment

Cramér-Rao lower bound for general case:

$$\mathbf{E}\left\{(\hat{x}^n(Z^n) - x(t^n;\theta))\,(\hat{x}^n(Z^n) - x(t^n;\theta))^{\mathsf{T}}\right\} \succcurlyeq$$

$$\succcurlyeq \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right) I(\theta)^{-1} \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right)^{\mathsf{T}} +$$

$$+ \,(\bar{x}^n(\theta) - x(t^n;\theta))\,(\bar{x}^n(\theta) - x(t^n;\theta))^{\mathsf{T}},$$

$$\min_{\hat{x}^n \in \mathrm{Equiv}} \mathbf{E}\left\{(\hat{x}^n(Z^n) - x(t^n;\theta))\,(\ldots)^{\mathsf{T}}\right\} \not\succcurlyeq$$

$$\not\succcurlyeq \min_{\substack{\hat{x}^n \in \mathrm{Equiv} \\ \bar{x}^n = \mathbf{E}\{\hat{x}^n\}}} \left(\left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right) I(\theta)^{-1} (\ldots)^{\mathsf{T}} + (\bar{x}^n(\theta) - x(t^n;\theta))\,(\ldots)^{\mathsf{T}}\right) \not\succcurlyeq$$

$$\not\succcurlyeq \min_{\substack{q \in \mathbb{R}^2, Q \in \mathbb{R}^{2\times d} \\ q, Q:\ \hat{x}^n \in \mathrm{Equiv}}} \left(Q\, I(\theta)^{-1} Q^{\mathsf{T}} + (q - x(t^n;\theta))\,(q - x(t^n;\theta))^{\mathsf{T}}\right)$$

Equivariant Estimates. Accuracy Assessment

Cramér-Rao lower bound for general case:

$$\mathbf{E}\left\{(\hat{x}^n(Z^n) - x(t^n;\theta))\,(\hat{x}^n(Z^n) - x(t^n;\theta))^{\mathsf{T}}\right\} \succcurlyeq$$

$$\succcurlyeq \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right) I(\theta)^{-1} \left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right)^{\mathsf{T}} +$$

$$+ (\bar{x}^n(\theta) - x(t^n;\theta))\,(\bar{x}^n(\theta) - x(t^n;\theta))^{\mathsf{T}},$$

$$\min_{\hat{x}^n \in \mathrm{Equiv}} \mathrm{tr}\left\{\mathbf{E}\left\{(\hat{x}^n(Z^n) - x(t^n;\theta))\,(\ldots)^{\mathsf{T}}\right\}\right\} \geqslant$$

$$\geqslant \min_{\substack{\hat{x}^n \in \mathrm{Equiv} \\ \bar{x}^n = \mathbf{E}\{\hat{x}^n\}}} \mathrm{tr}\left\{\left(\frac{\partial \bar{x}^n(\theta)}{\partial \theta}\right) I(\theta)^{-1} (\ldots)^{\mathsf{T}} + (\bar{x}^n(\theta) - x(t^n;\theta))\,(\ldots)^{\mathsf{T}}\right\} \geqslant$$

$$\geqslant \min_{\substack{q \in \mathbb{R}^2, Q \in \mathbb{R}^{2\times d} \\ q, Q:\ \hat{x}^n \in \mathrm{Equiv}}} \mathrm{tr}\left\{Q\, I(\theta)^{-1} Q^{\mathsf{T}} + (q - x(t^n;\theta))\,(q - x(t^n;\theta))^{\mathsf{T}}\right\}$$

Equivariant Estimates. Accuracy Assessment

$$\min_{\hat{x}^n \in \mathrm{Equiv}} \mathrm{tr}\Big\{ \mathbf{E}\Big\{ (\hat{x}^n(Z^n) - x(t^n;\theta))\,(\ldots)^{\mathsf{T}} \Big\}\Big\} =$$

$$= \min_{\hat{x}^n \in \mathrm{Equiv}} \mathbf{E}\Big\{ (\hat{x}^n(Z^n) - x(t^n;\theta))^2 \Big\} \geqslant$$

$$\geqslant \min_{\substack{q \in \mathbb{R}^2, Q \in \mathbb{R}^{2\times d} \\ q,Q:\ \hat{x}^n \in \mathrm{Equiv}}} \mathrm{tr}\Big\{ Q\, I(\theta)^{-1} Q^{\mathsf{T}} \Big\} + (q - x(t^n;\theta))^2 > 0 \;!!!\,.$$

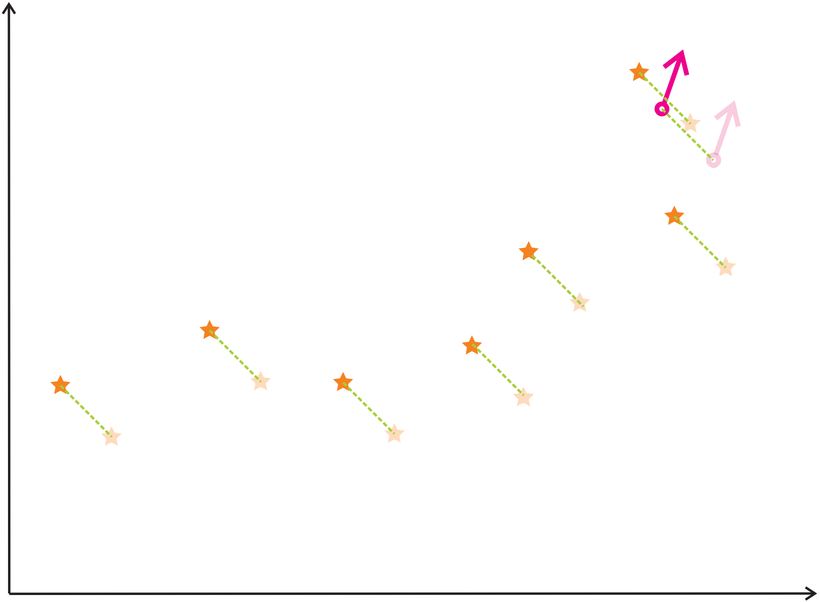
Equivariant Estimates. Accuracy Assessment

$$\min_{\hat{x}^n \in \mathrm{Equiv}} \mathbf{E}\Big\{ \big(\hat{x}^n(Z^n) - x(t^n;\theta)\big)^2 \Big\} \geqslant$$

$$\geqslant \min_{\substack{q\in\mathbb{R}^2, Q\in\mathbb{R}^{2\times d} \\ q,Q:\ \hat{x}^n\in\mathrm{Equiv}}} \mathrm{tr}\Big\{ Q\, I(\theta)^{-1} Q^{\mathsf{T}} \Big\} + \big(q - x(t^n;\theta)\big)^2 \ .$$

$$\begin{cases} \mathrm{tr}\big\{ Q\, I(\theta)^{-1} Q^{\mathsf{T}} \big\} + \big(q - x(t^n;\theta)\big)^2 \to \min, \\ Q\dot{T}_0^k(\theta) = \dot{B}_0^k q + \dot{b}_0^k \,, \quad k \in 1, \ldots, K. \end{cases}$$

Equivariant Est. Transformation on the Plane



Equivariant Est. Transformation on the Plane

Translation along Ox^n:

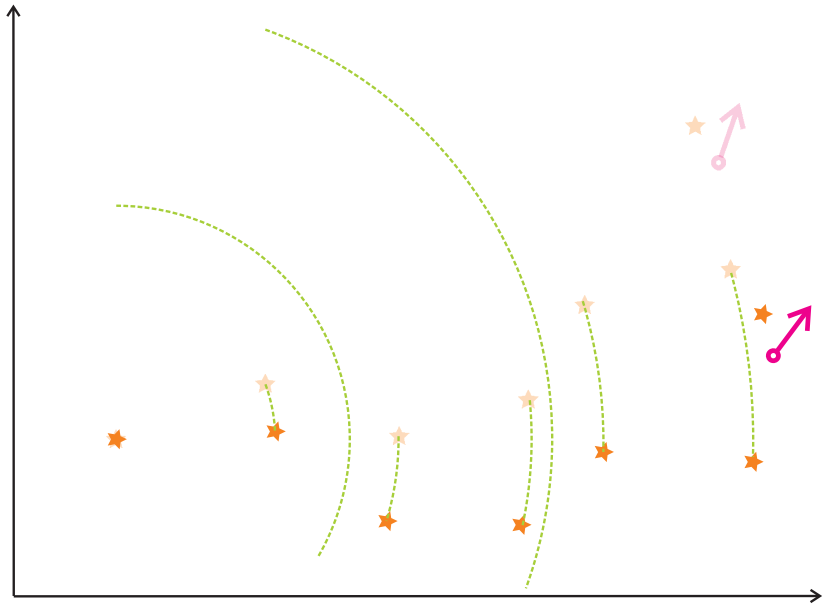
$$s_p(x) = x + \begin{bmatrix} p \\ 0 \end{bmatrix}$$

$$\dot{B}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \dot{b}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T_p(\theta) = T_p\left(\begin{bmatrix} x_{N0} & x_{E0} & \varphi_0 & v_0 & a_0^{ort} & a_0^{tng} & t_1 & \dots \end{bmatrix}^\mathsf{T}\right) =$$
$$= \begin{bmatrix} x_{N0} + p & x_{E0} & \varphi_0 & v_0 & a_0^{ort} & a_0^{tng} & t_1 \dots \end{bmatrix}^\mathsf{T},$$

$$\dot{T}_0(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \dots \end{bmatrix}^\mathsf{T}$$

Equivariant Est. Transformation on the Plane



Equivariant Est. Transformation on the Plane

Rotation:

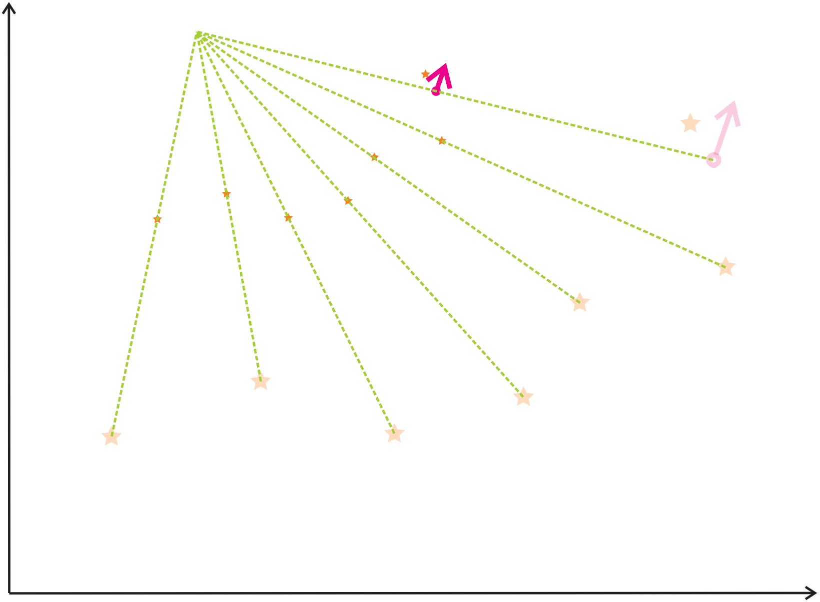
$$s_p(x) = \begin{bmatrix} \cos(p) & -\sin(p) \\ \sin(p) & \cos(p) \end{bmatrix} x$$

$$\dot{B}_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \dot{b}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_p(\theta) = T_p\left(\begin{bmatrix} [x_{N0}\, x_{E0}]^\mathsf{T} & \varphi_0 & v_0 & a_0^{ort} & a_0^{tng} & t_1 & \ldots \end{bmatrix}^\mathsf{T}\right) =$$
$$= \begin{bmatrix} [x_{N0}\, x_{E0}]^\mathsf{T} B_p & \varphi_0 + p & v_0 & a_0^{ort} & a_0^{tng} & t_1 \ldots \end{bmatrix}^\mathsf{T},$$

$$\dot{T}_0(\theta) = \begin{bmatrix} [x_{N0}\, x_{E0}]^\mathsf{T} \dot{B}_0 & 1 & 0 & 0 & 0 & 0 \ldots \end{bmatrix}^\mathsf{T}$$

Equivariant Est. Transformation on the Plane



Equivariant Est. Transformation on the Plane

Scale:

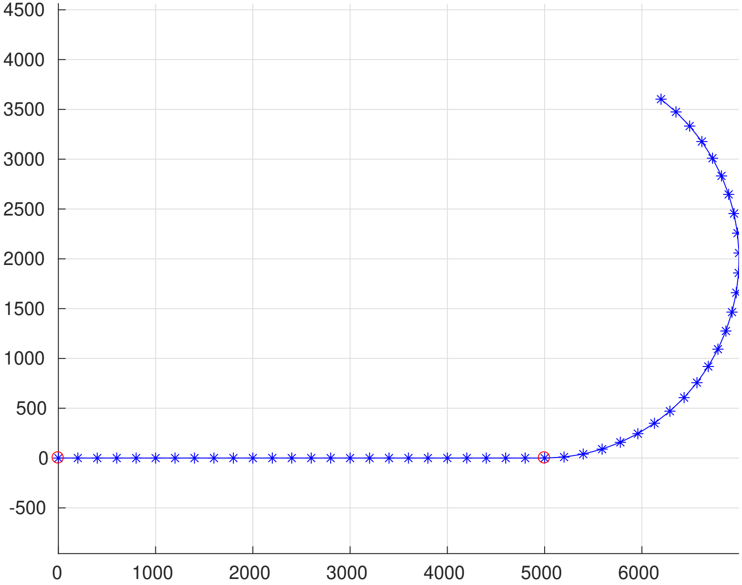
$$s_p(x) = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} x$$

$$\dot{B}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \dot{b}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

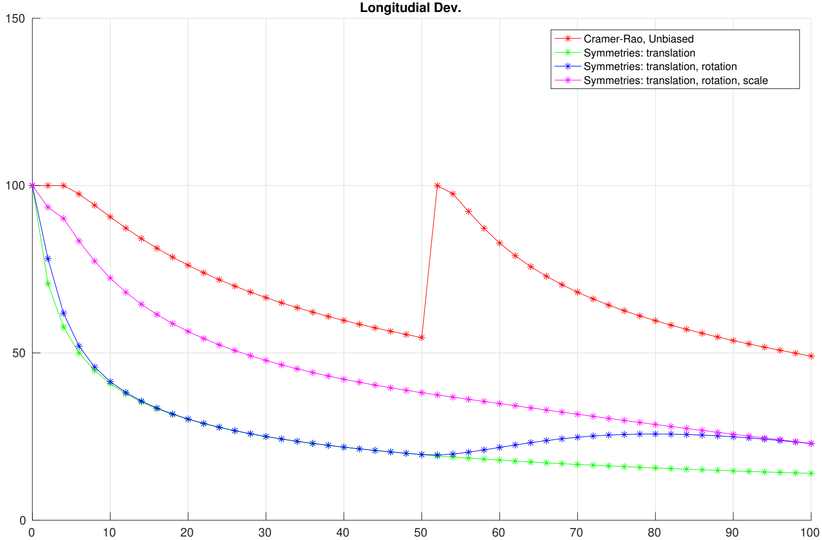
$$T_p(\theta) = T_p\left(\begin{bmatrix} x_{N0} & x_{E0} & \varphi_0 & v_0 & a_0^{ort} & a_0^{tng} & t_1 & \dots \end{bmatrix}^\mathsf{T}\right) =$$
$$= \begin{bmatrix} px_{N0} & px_{E0} & \varphi_0 & pv_0 & pa_0^{ort} & pa_0^{tng} & t_1 \dots \end{bmatrix}^\mathsf{T},$$

$$\dot{T}_0(\theta) = \begin{bmatrix} x_{N0} & x_{E0} & 0 & v_0 & a_0^{ort} & a_0^{tng} & 0 \dots \end{bmatrix}^\mathsf{T}$$

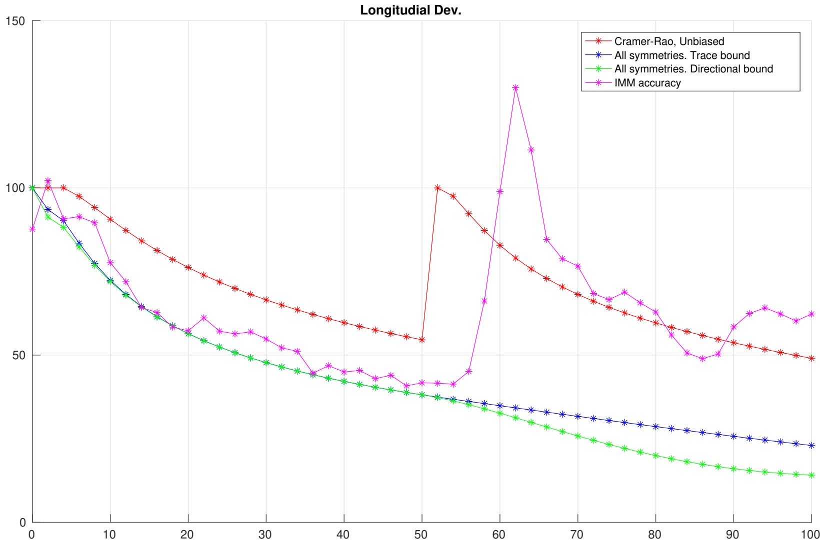
Equivariant Est. Transformation on the Plane



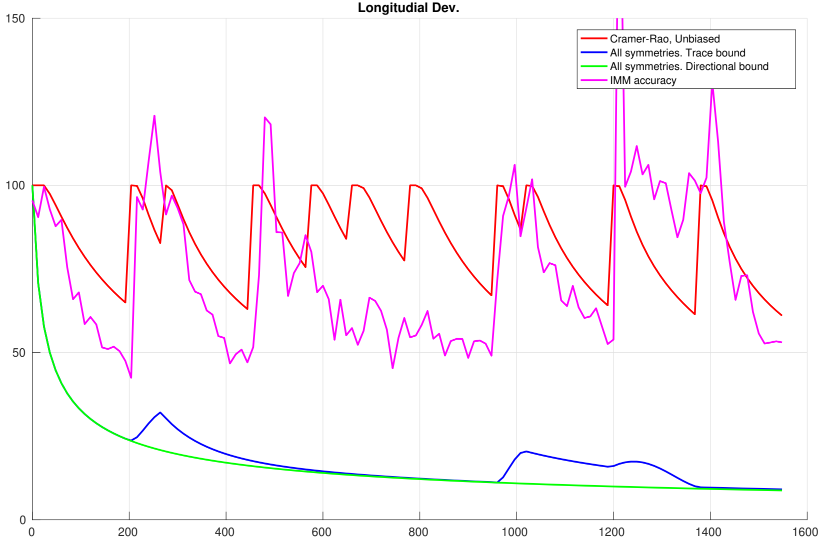
Equivariant Est. Transformation on the Plane



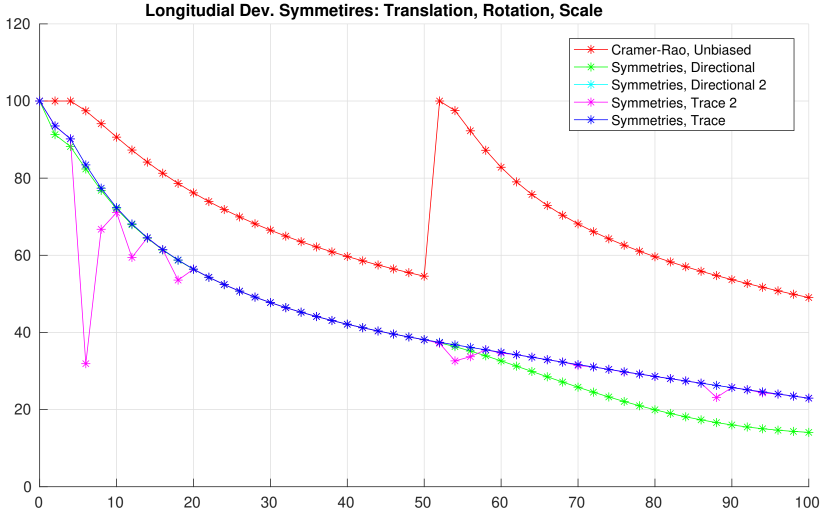
Equivariant Est. Transformation on the Plane



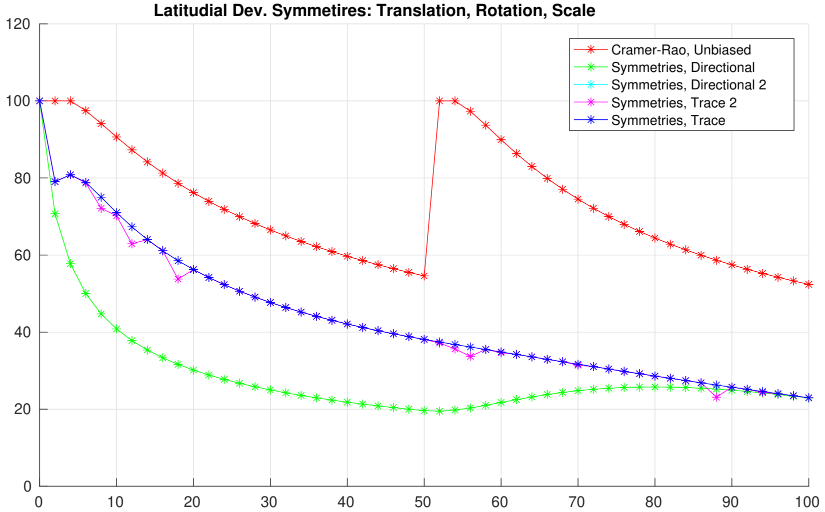
Equivariant Est. Transformation on the Plane



Equivariant Estimates. Attempt to Improve



Equivariant Estimates. Attempt to Improve



Linear Equivariant Estimates

The class of linear estimates $\mathcal{K}_l$

$$\hat{x}^n(Z^n) = \sum_{i=1}^{n} L_i z^i = LZ^n \, .$$

The class of linear equivariant estimates $\mathcal{K}_{el}(\mathcal{X}^{\theta}_{switch})$:

$$\mathcal{K}_{el}(\mathcal{X}^{\theta}_{switch}) = \mathcal{K}_l \cap \mathcal{K}_e(\mathcal{X}^{\theta}_{switch}).$$

Coincidence

Theorem

If $W^i = \sigma^2 I \quad (i = 1, \ldots, n)$, and $n \geqslant N \geqslant 2$,
and the class $\mathcal{X}^\theta_{switch}$ admits the scale transformation
(with $\dot{B}_0 = qI$, $q > 0$, $\dot{b}_0 = 0$)
then

$$\inf_{\hat{x}_i \in \mathcal{K}_{el}(\mathcal{X}^\theta_{switch})} \mathbf{E}\left\{\|\hat{x}_i(Z_i^x) - x(t_i)\|^2\right\} = \\ = \inf_{\hat{x}_i \in \mathcal{K}_{e}(\mathcal{X}^\theta_{switch})} \mathbf{E}\left\{\|\hat{x}_i(Z_i^x) - x(t_i)\|^2\right\}.$$