Feedback strategies for controlled continuity equation

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Statement of the problem

Consider the multiagent system with the dynamics of each agent given by

$$\dot{x}(t) = f(t, x(t), m(t), u(t, m(t))), \ x(s) = x_0, \ m(s) = m_0.$$

Here $t \in [0, T]$, $x \in \mathbb{T}^d$, $m(t) \in P^2(\mathbb{T}^d)$, $u \in U \subset \mathbb{R}^n$ is a compact, $s \in [0, T]$.
Here and below $\mathbb{T}^d$ denotes the d-dimensional torus: $\mathbb{T}^d \triangleq \mathbb{R}^d/\mathbb{Z}^d$, when $\mathcal{P}^2(\mathbb{T}^d)$ stands for the set of probabilities on $\mathbb{T}^d$ with the finite second moment. We endow this space with the second Wasserstein metric W_2.

Continuity equation

The dynamics of the whole system obeys the continuity equation

$$\frac{\partial m(t)}{\partial t} + \operatorname{div}\left(f(t, \cdot, m(t), u(t, m(t)))m(t)\right) = 0, \quad m(s) = m_0.$$

We assume that the all agent are influenced by the same control depending only on current time and current distribution of agents $m(t)$.

Continuity equation

The payoff functional is equal to

$$J(t_0, m_0, u) = g(m(T)).$$

We maximize this functional on the set of admissible controls.

Assumptions

A1 The function f is continuous. There exists the constant $C_0: \ \forall t \in [0, T], \ x \in T^d, \ m(t) \in \mathcal{P}^2(\mathbb{T}^d), \ u \in U$ the following inequality is valid

$$||f(t, x, m, u)|| \leq C_0.$$

A2 $||f(t, x, m, u) - f(t', x, m, u)|| \leq w_f(t - t'), \forall t \in [0, T]$, $x \in T^d$. **A3** $|g(m) - g(m')| \leq w_g(W_2(m, m')), \ m, m' \in \mathcal{P}^2(\mathbb{T}^d)$, $u \in U$. **A4** There exists constant $L > 0$: $\forall t \in [0, T], \ x, x' \in T^d$, $m, m' \in \mathcal{P}^2(\mathbb{T}^d), \ u \in U$

$$||f(t, x, m, u) - f(t, x', m', u)|| \leq L(||x - x'|| + W_2(m, m')).$$

In conditions **A2**, **A3** ω_f and ω_g are modules of continuity. Denote $\tilde{U}$ the set of relaxed controls.

u-stable function

Definition. We say that a lower semicontinuous function $\psi : [0, T] \times \mathcal{P}^2(\mathbb{T}^d) \to \mathbb{R}$ is *u-stable* if

- for any $m \in \mathcal{P}^2(\mathbb{T}^d)$, $g(m) \leq \psi(T, m)$;
- for any $s, r \in [0, T]$, $s < r$, $m_0 \in \mathcal{P}^2(\mathbb{T}^d)$, there exists a relaxed control $\xi : \psi(s, m_0) \geq \psi(r, m(r, s, m_0, \xi))$.

Additional assumptions

Consider the function $\rho(\varepsilon, t) : \lim_{\varepsilon \to 0} \rho(\varepsilon, t) = 0$. Let $(s, m) \in [0, T] \times \mathcal{P}^2(\mathbb{T}^d)$, $\nu \in \mathcal{P}^2(\mathbb{T}^d)$ be such that

$$W_2^2(m, \nu) \leq \rho(\varepsilon, s),$$

$$\psi(s, \nu) = \min\{\psi(s, m') : m' \in \mathcal{P}^2(\mathbb{T}^d),\ W_2^2(m, m') \leq \rho(\varepsilon, s)\}.$$

Here ψ is a u-stable function.
We shall suppose that

$$W_2(m(r), m(s)) \leq C_0(r - s)\ \forall m(\cdot).$$

Main result

Theorem. Let $s, r \in [0, T]$, $s \leq r$, $m_0, \nu_0 \in \mathcal{P}^2(\mathbb{T}^d)$, π is an optimal plan between m_0, ν_0; $m(\cdot) = m(\cdot, s, m_0, u)$; $\nu(\cdot) = \nu(\cdot, s, \nu_0, \eta)$. Then

$$W_2^2(m(r), \nu(r)) \leq W_2^2(m_0, \nu_0)(1 + 4L(r-s)) + 4C_0^2(r-s)^2 +$$

$$4\sqrt{d}(w_f(r-s) + LC_0(r-s)^2).$$

Thank you for attention!