Abstract fractional McKean-Vlasov and HJB equations

Marianna Troeva[1]
(Joint work with Vassili N. Kolokoltsov[2])

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[1]North-Eastern Federal University, Yakutsk, Russia
[2]Dep. of Statistics, University of Warwick, Coventry CV4 7AL, UK

Outline

Introduction

In this work we introduce and study a new class of abstract nonlinear fractional pseudo-differential equations in Banach spaces. These abstract equations include both the McKean-Vlasov-type equations describing nonlinear Markov processes and the Hamilton-Jacobi-Bellman-Isaacs equations of stochastic control and games thus allowing for a unified analysis of these equations.

We prove the well-posedness results for these abstract equations, and their continuous dependence on the initial data. Our approach is based on the mild solutions to the fractional nonlinear equations that are based on the Zolotarev integral representation for the Mittag-Leffler functions.

Introduction

We study the nonlinear Cauchy problems of the form

$$\dot{b}(t) = Ab(t) + H(t, b(t), Db(t), \alpha), \quad b(a) = Y, \quad t \geq a, \tag{1}$$

where A, $D_1, \cdots, D_n$ are unbounded linear operators in a Banach space B, $D = (D_1, \cdots, D_n)$, α is a parameter from another Banach space B^{par} and H is a continuous mapping $\mathbf{R} \times B \times B^n \times B^{par} \to B$, and their fractional counterparts

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which are the following Cauchy problems

$$D^{\beta}_{a+*}b(t) = Ab(t) + H(t, b(t), Db(t), \alpha), \quad b(a) = Y \quad t \geq a, \tag{2}$$

where D^{β}_{a+*} is the Caputo-Dzherbashyan (CD) fractional derivative of order $\beta \in (0, 1)$,

$$D^{\beta}_{a+*}b(t) = \frac{1}{\Gamma(-\beta)} \int_0^{t-a} \frac{b(t-z) - b(t)}{z^{1+\beta}} dz + \frac{b(t) - b(a)}{\Gamma(1-\beta)(t-a)^{\beta}}. \tag{3}$$

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Our main examples concern the case when B is a space of functions on $\mathbf{R}^d$ and D is the gradient (derivative) operator.
• The fractional Hamilton-Jacobi-Bellmann-Isaacs equation of controlled Markov processes (with an external parameter) is the equation of the form

$$D^{\beta}_{a+*}f(t,x) = Af(t,x) + H(t,x,f(t,x),\frac{\partial f}{\partial x}(t,x),\alpha), \quad (4)$$

for which the most natural Banach space is $B = C_\infty(\mathbf{R}^d)$ (the space of bounded continuous functions on $\mathbf{R}^d$ tending to zero at infinity).

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The Hamiltonian H arising from optimal control usually even does not depend explicitly on f, just on its gradient, and it writes down as

$$H(t,x,f,p,\alpha) = H(t,x,p,\alpha) = \sup_{u \in U}[J(t,x,u) + g(t,x,u)p], \tag{5}$$

with some functions J, g, where U is a compact set of controls (or with $\inf\sup$ instead of just $\sup$ in case of Isaacs equations). For such H the fractional equation (4) was derived in (V. Kolokoltsov and M. Veretennikova, 2014) as a Bellman equation for optimal control of scaled limits of continuous time random walks.

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• The fractional version of the McKean-Vlasov type equations (describing nonlinear Markov processes in the sense of (V. Kolokoltsov, Cambridge Univ. Press, 2010)) are the quasi-linear equations of type

$$D^{\beta}_{a+*}f(t,x) = A^* f(t,x) + \sum_{j=1}^{d} h_j(t,x,\{f(t,.)\},\alpha)\frac{\partial f}{\partial x_j}(t,x), \tag{6}$$

for which the most natural Banach space is $L_1(\mathbf{R}^d)$ (or the space of Borel measures on $\mathbf{R}^d$). In these equations A is the generator of a Feller process in $\mathbf{R}^d$ and A^* is its dual operators.

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While H in (4) depends on the point-wise values of f, the functions h_j in (6) usually depend on some integrals of f. The abstract framework of equations (2) allows one to treat these cases in a unified way, as well as to include in the theory in a more or less straightforward way important new cases, for instance, fractional HJB-Isaacs or McKean-Vlasov-type equations on manifolds.

Preliminaries

By $E_\beta(x)$ we denote the standard Mittag-Leffler function of index β:

$$E_\beta(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta k + 1)}.$$

For our purpose the most convenient formula for the Mittag-Leffler function is its integral representation (Zolotarev formula, or Zolotarev-Pollard formula)

$$E_\beta(s) = \frac{1}{\beta} \int_0^\infty e^{sx} x^{-1-1/\beta} G_\beta(1, x^{-1/\beta})\, dx, \qquad (7)$$

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where

$$G_\beta(t,x) = \frac{1}{\pi} Re \int_0^\infty \exp\left\{ipx - tp^\beta e^{i\pi\beta/2}\right\} dp$$

is the heat kernel (solution with the Dirac initial condition) of the equation

$$\frac{\partial G}{\partial t}(t,x) = -\frac{d^\beta}{dx^\beta} G(t,x),$$

or, in probabilistic terms, the transition probability density of the stable Lévy subordinator of index β.
The convenience of the formula (7) is due to the fact that it allows one to define $E_\beta(A)$ for an operator A whenever A generates a semigroup, so that e^{At} is well defined.

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For a Banach space B and $\tau < t$ we denote $C([\tau, t], B)$ the Banach space of continuous functions $f : [\tau, t] \to B$ with the norm

$$\|f\|_{C([\tau,t],B)} = \sup_{s \in [\tau,t]} \|f(s)\|_B,$$

and $C_Y([\tau, t], B)$ its closed subset consisting of functions f such that $f(\tau) = Y$, which is a complete metric space under the induced topology.
For a closed convex subset M of B, $C_Y([\tau, t], M)$ denotes a convex subset of $C_Y([\tau, t], B)$ of functions with values in M.

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Let us recall Theorem 2.1.3 (Kolokoltsov, Birkhäuser, 2019). It is a version of the fixed-point principle, specifically tailored to be used for nonlinear diffusions and fractional equations.

Theorem 1. Suppose that for any $Y \in M$, $\alpha \in B_1$, with B_1 another Banach space, a mapping $\Phi_{Y,\alpha} : C([\tau, T], M) \to C_Y([\tau, T], M)$ is given with some $T > \tau$ such that for any t the restriction of $\Phi_{Y,\alpha}(\mu_.)$ on $[\tau, t]$ depends only on the restriction of the function μ_s on $[\tau, t]$. Moreover,

$$\|[\Phi_{Y,\alpha}(\mu_.^1)](t) - [\Phi_{Y,\alpha}(\mu_.^2)](t)\|$$
$$\leq L(Y) \int_\tau^t (t-s)^{-\omega} \|\mu_.^1 - \mu_.^2\|_{C([\tau,s],B)}\, ds, \qquad (8)$$

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$$\|[\Phi_{Y_1,\alpha_1}(\mu_\cdot)](t)-[\Phi_{Y_2,\alpha_2}(\mu_\cdot)](t)\| \le \varkappa\|Y_1-Y_2\|+\varkappa_1\|\alpha_1-\alpha_2\|,$$

for all $t \in [\tau, T]$, $\mu^1, \mu^2 \in C([\tau, T], M)$, $\alpha_1, \alpha_2 \in B_1$, some constants $\varkappa, \varkappa_1 \ge 0$, $\omega \in [0, 1)$, and a continuous function L on M.
Then for any $Y \in M$, $\alpha \in B_1$ the mapping $\Phi_{Y,\alpha}$ has a unique fixed point $\mu_{t,\tau}(Y, \alpha)$ in $C_Y([\tau, T], M)$.
Moreover, if $\omega > 0$, then for all $t \in [\tau, T]$,

$$\|\mu_{t,\tau}(Y, \alpha) - Y\|$$

$$\le E_{1-\omega}(L(Y)\Gamma(1-\omega)(t-\tau)^{1-\omega})\|[\Phi_{Y,\alpha}(Y)](t) - Y\|, \quad (9)$$

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and the fixed points $\mu_{t,\tau}(Y_1, \alpha_1)$ and $\mu_{t,\tau}(Y_2, \alpha_2)$ with different initial data Y_1, Y_2 and parameters α_1, α_2 enjoy the estimate (for any $j = 1, 2$)

$$\|\mu_{t,\tau}(Y_1, \alpha_1) - \mu_{t,\tau}(Y_2, \alpha_2)\|$$

$$\leq (\varkappa\|Y_1 - Y_2\| + \varkappa_1\|\alpha_1 - \alpha_2\|)E_{1-\omega}(L(Y_j)\Gamma(1-\omega)(t-\tau)^{1-\omega}). \tag{10}$$

If $\omega = 0$, these estimates are simplified to

$$\|\mu_{t,\tau}(Y, \alpha) - Y\| \leq e^{(t-\tau)L(Y)}\|[\Phi_{Y,\alpha}(Y)](t) - Y\|, \tag{11}$$

$$\|\mu_{t,\tau}(Y_1, \alpha_1) - \mu_{t,\tau}(Y_2, \alpha_2)\|$$

$$\leq (\varkappa\|Y_1 - Y_2\| + \varkappa_1\|\alpha_1 - \alpha_2\|)\exp\{(t-\tau)\min(L(Y_1), L(Y_2))\}. \tag{12}$$

Well-posedness

For two Banach spaces B, C we denote by $\mathcal{L}(B, C)$ the Banach space of bounded linear operators $B \to C$ with the usual operator norm denoted $\|.\|_{B \to C}$.
The sequences of embedded Banach spaces $B_2 \subset B_1 \subset B$ with the norms denoted $\|.\|_2, \|.\|_1, \|.\|$ respectively, will be referred to as the *Banach triple* (of inserted spaces) or a *Banach tower of order* 3, if the norms are ordered, $\|.\|_2 \geq \|.\|_1 \geq \|.\|$, and B_2 is dense in B_1 in the topology of B_1 while B_1 is dense in B in the topology of B.

Well-posedness

The following setting will play the key role in this work.
Conditions (A):
(i) Let $B_2 \subset B_1 \subset B$ be the Banach triple, with the norms denoted $\|.\|_2, \|.\|_1, \|.\|$ respectively, and let

$$D_i \in \mathcal{L}(B_1, B) \cap \mathcal{L}(B_2, B_1), \quad i = 1, \cdots, n.$$

Without loss of generality we assume that norms of all D_j are bounded by 1 in both $\mathcal{L}(B_1, B)$ and $\mathcal{L}(B_2, B_1)$ (which is usually the case in applications below).
(ii) Let $A \in \mathcal{L}(B_2, B)$ and let A generate a strongly continuous semigroup e^{At} in both B and B_1, so that

$$\|e^{At}\|_{B \to B} \leq Me^{mt}, \quad \|e^{At}\|_{B_1 \to B_1} \leq M_1 e^{tm_1}, \qquad (13)$$

with some nonnegative constants M, m, M_1, m_1, and B_2 is an invariant core for this semigroup in B.

Well-posedness

(iii) Let B^{par} be another Banach space (of parameters) with the norm denoted $\|.\|_{par}$ and $H : \mathbf{R} \times B \times B^n \times B^{par} \to B$ be a continuous mapping, which is Lipschitz in the sense that

$$\|H(t, b_0, b_1, \cdots, b_n, \alpha) - H(t, \tilde{b}_0, \tilde{b}_1, \cdots, \tilde{b}_n, \alpha)\|$$

$$\leq L_H \sum_{j=0}^{n} \|b_j - \tilde{b}_j\|(1 + L'_H \sum_{j=1}^{n} \|b_j\|), \tag{14}$$

$$\|H(t, b_0, b_1, \cdots, b_n, \alpha) - H(t, b_0, b_1, \cdots, b_n, \tilde{\alpha})\|$$

$$\leq L_H^{par} \|\alpha - \tilde{\alpha}\|_{par}(1 + \sum_{j=0}^{n} \|b_j\|), \tag{15}$$

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and is of linear growth

$$\|H(t, b_0, b_1, \cdots, b_n, \alpha)\| \leq L_H \sum_{j=0}^{n} \|b_j\|, \quad (16)$$

with some constants L_H, L'_H, L_H^{par}.

Remark 1. *For the classical HJB equations estimate* (14) *holds with* $L'_H = 0$*, in which case* (16) *follows from* (14)*. However, for McKean-Vlasov-type equations the linear growth of the Lipschitz constant in* (14) *is not avoidable.*

Well-posedness

An important assumption in our analysis is the following smoothing property of the semigroup e^{At}: for $t > 0$ it takes B to B_1 and

$$\|e^{At}\|_{B \to B_1} \le \varkappa t^{-\omega}, \quad t \in (0, 1], \tag{17}$$

with some constants $\varkappa > 0$ and $\omega \in (0, 1)$. Sometimes a similar condition for the pair B_1, B_2 is used:

$$\|e^{At}\|_{B_1 \to B_2} \le \varkappa_1 t^{-\omega}, \quad t \in (0, 1]. \tag{18}$$

Remark 2. *For pseudo-differential operators A (including the generators of Feller semigroups) the deeper smoothing property* (18) *can be derived from* (17) *and the smoothness of the symbol of A, see Theorem 5.15.1 in (V.Kolokoltsov, Birkhäuser, 2019).*

Well-posedness

The so-called mild version of the Cauchy problem (1) is the integral equation

$$b(t) = e^{A(t-a)}Y + \int_a^t e^{A(t-s)} H(s, b(s), Db(s), \alpha)\, ds, \qquad t \geq a. \tag{19}$$

It is well known that if $b(t)$ solves equation (1), then it solves also equation (19), so that the uniqueness for (19) implies the uniqueness for (1).

Well-posedness

Theorem 2. *Let conditions (A) and smoothing property* (17) *hold. Then equation* (19) *is well posed in* B_1*, that is, for any* $Y \in B_1$, $\alpha \in B^{par}$ *there exists its unique global solution* $b(t) = b(t; Y, \alpha) \in B_1$*, which depends Lipschitz continuously on the initial data* Y *and the parameter* α*. In particular,*

$$\sup_{t\in[a,T]} \|b(t; Y, \alpha) - b(t; \tilde{Y}, \tilde{\alpha})\|_1$$

$$\leq K \left(\|\alpha - \tilde{\alpha}\|_{par}(1 + \|Y\|_1) + \|Y - \tilde{Y}\|_1 \right), \qquad (20)$$

with constant K *depending on* $t - a$ *and all constants entering the assumptions of the theorem.*

Well-posedness

To treat equation (2) we recall that its mild form is the integral equation

$$b(t) = E_\beta(A(t-a)^\beta)Y$$

$$+\beta \int_a^t (t-s)^{\beta-1} E'_\beta(A(t-s)^\beta) H(s, b(s), Db(s), \alpha)\, ds, \quad (21)$$

where $E_\beta(A)$ is defined by (7). Thus more explicitly this equation writes down as

$$b(t) = \frac{1}{\beta} \int_0^\infty e^{A(t-a)^\beta x} Y x^{-1-1/\beta} G_\beta(1, x^{-1/\beta})\, dx$$

$$+\int_a^t (t-s)^{\beta-1} \int_0^\infty e^{A(t-s)^\beta x} x^{-\frac{1}{\beta}} G_\beta(1, x^{-\frac{1}{\beta}})\, dx H(s, b(s), Db(s), \alpha) ds \quad (22)$$

Well-posedness

Theorem 3. *Let conditions (A) and smoothing* (17) *hold. Then equation* (21) *is well posed in* B_1, *that is, for any* $Y \in B_1$ *there exists its unique global solution* $b(t) \in B_1$, *which depends Lipschitz continuously on the initial data* Y *and parameter* α *so that* (20) *holds.*

Well-posedness

Remark 3. *All the results above have their straightforward counterparts for the backward problems under the same exact conditions. The only difference is that, instead of the sets* $C_Y([\tau, t], B)$ *of functions with a fixed left point, in the backward setting one is working with the sets* $C_Z^b([\tau, T], B)$ *of functions with a fixed right point:* $f(T) = Z$.

Well-posedness

Let us comment on how the results above are applied to HJB and McKean-Vlasov equations (4) and (6). For these cases, B is a space of functions on $\mathbf{R}^d$, A is the generator of a Feller process in R^d and D is the gradient (derivative) operator. Specifically, for HJB equation the natural Banach triple is
$C^2_\infty(\mathbf{R}^d) \subset C^1_\infty(\mathbf{R}^d) \subset C_\infty(\mathbf{R}^d)$,
and for the McKean-Vlasov equations a possible triple is
$W_2(\mathbf{R}^d) \subset W_1(\mathbf{R}^d) \subset L_1(\mathbf{R}^d)$.

Well-posedness

Here $C_\infty(\mathbf{R}^d)$ denotes the space of continuous functions on $\mathbf{R}^d$ tending to zero at infinity, $C^j_\infty(\mathbf{R}^d)$ their subsets of functions having derivatives of order up to j in $C_\infty(\mathbf{R}^d)$, and $W_j(R^d)$ denote the Sobolev spaces of functions with partial derivatives (understood in the sense of generalized functions) of order up to j in $L_1(\mathbf{R}^d)$, equipped with the integral (Sobolev) norms

$$\|f\|_{L_1(R^d)} = \int |f(x)|\, dx, \quad \|f\|_{W_1(R^d)} = \|f\|_{L_1(R^d)} + \sum_{j=1}^d \int \left|\frac{\partial f}{\partial x_j}\right| dx,$$

$$\|f\|_{W_2(R^d)} = \|f\|_{W_1(R^d)} + \sum_{i\le j} \int \left|\frac{\partial^2 f}{\partial x_j \partial x_i}\right| dx.$$

Conclusions

- We introduced a class of abstract nonlinear fractional pseudo-differential equations in Banach spaces that includes both Mc-Kean-Vlasov type equations and HJB equations thus allowing for a unified analysis of these equations.
- Looking at these equations as evolving in dual Banach triples allows us to recast directly the properties of one type to the properties of another type leading to an effective theory of coupled forward-backward systems (forward McKean-Vlasov evolution and backward HJB evolution) which are central to the modern theory of mean-field games.
- We prove the well-posedness results for these abstract equations, and their continuous dependence on the initial data.
- Our result on coupled forward-backward systems of nonlinear fractional equations is a step to the study of fractional MFG on manifolds including quantum fractional MFG.

Thank you for your attention!!